CHAPTER 6

Some Methods in the Calculus of Variations

6.1 Introduction

Many problems in Newtonian mechanics are more easily analyzed by means of alternative statements of the laws, including **Lagrange's equation** and **Hamilton's principle.*** As a prelude to these techniques, we consider in this chapter some general principles of the techniques of the calculus of variations.

Emphasis will be placed on those aspects of the theory of variations that have a direct bearing on classical systems, omitting some existence proofs. Our primary interest here is in determining the path that gives extremum solutions, for example, the shortest distance (or time) between two points. A well-known example of the use of the theory of variations is **Fermat's principle:** Light travels by the path that takes the least amount of time (see Problem 6-7).

6.2 Statement of the Problem

The basic problem of the calculus of variations is to determine the function $y(x)$ such that the integral

$$J = \int_{x_1}^{x_2} f\{y(x), y'(x); x\} \, dx \tag{6.1}$$

*The development of the calculus of variations was begun by Newton (1686) and was extended by Johann and Jakob Bernoulli (1696) and by Euler (1744). Adrien Legendre (1786), Joseph Lagrange (1788), Hamilton (1833), and Jacobi (1837) all made important contributions. The names of Peter Dirichlet (1805–1859) and Karl Weierstrass (1815–1879) are particularly associated with the establishment of a rigorous mathematical foundation for the subject.

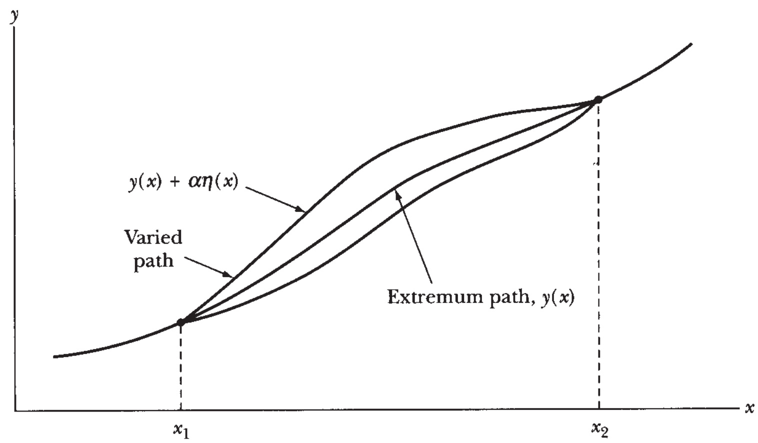

FIGURE 6-1 The function $y(x)$ is the path that makes the functional J an extremum. The neighboring functions $y(x) + \alpha\eta(x)$ vanish at the endpoints and may be close to $y(x)$, but are not the extremum.

is an *extremum* (i.e., either a maximum or a minimum). In Equation 6.1, $y'(x) \equiv dy/dx$, and the semicolon in f separates the independent variable x from the dependent variable $y(x)$ and its derivative $y'(x)$. The functional* J depends on the function $y(x)$, and the limits of integration are fixed.† The function $y(x)$ is then to be varied until an extreme value of J is found. By this we mean that if a function $y = y(x)$ gives the integral J a minimum value, then any *neighboring function,* no matter how close to $y(x)$, must make J increase. The definition of a neighboring function may be made as follows. We give all possible functions y a parametric representation $y = y(\alpha, x)$ such that, for $\alpha = 0$, $y = y(0, x) = y(x)$ is the function that yields an extremum for J. We can then write

$$y(\alpha, x) = y(0, x) + \alpha\eta(x) \tag{6.2}$$

where $\eta(x)$ is some function of x that has a continuous first derivative and that vanishes at x_1 and x_2, because the varied function $y(\alpha, x)$ must be identical with $y(x)$ at the endpoints of the path: $\eta(x_1) = \eta(x_2) = 0$. The situation is depicted schematically in Figure 6-1.

If functions of the type given by Equation 6.2 are considered, the integral J becomes a functional of the parameter α:

$$J(\alpha) = \int_{x_1}^{x_2} f\{y(\alpha, x), y'(\alpha, x); x\}\, dx \tag{6.3}$$

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*The quantity J is a generalization of a function called a *functional,* actually an integral functional in this case.

†It is not necessary that the limits of integration be considered fixed. If they are allowed to vary, the problem increases to finding not only $y(x)$ but also x_1 and x_2 such that J is an extremum.

The condition that the integral have a *stationary value* (i.e., that an extremum results) is that J be independent of α in first order along the path giving the extremum ($\alpha = 0$), or, equivalently, that

$$\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} = 0 \tag{6.4}$$

for all functions $\eta(x)$. This is only a *necessary* condition; it is not sufficient.

EXAMPLE 6.1

Consider the function $f = (dy/dx)^2$, where $y(x) = x$. Add to $y(x)$ the function $\eta(x) = \sin x$, and find $J(\alpha)$ between the limits of $x = 0$ and $x = 2\pi$. Show that the stationary value of $J(\alpha)$ occurs for $\alpha = 0$.

Solution. We may construct neighboring varied paths by adding to $y(x)$,

$$y(x) = x \tag{6.5}$$

the sinusoidal variation $\alpha \sin x$,

$$y(\alpha, x) = x + \alpha \sin x \tag{6.6}$$

These paths are illustrated in Figure 6-2 for $\alpha = 0$ and for two different nonvanishing values of α. Clearly, the function $\eta(x) = \sin x$ obeys the endpoint conditions, that is, $\eta(0) = 0 = \eta(2\pi)$. To determine $f(y, y'; x)$ we first determine

$$\frac{dy(\alpha, x)}{dx} = 1 + \alpha \cos x \tag{6.7}$$

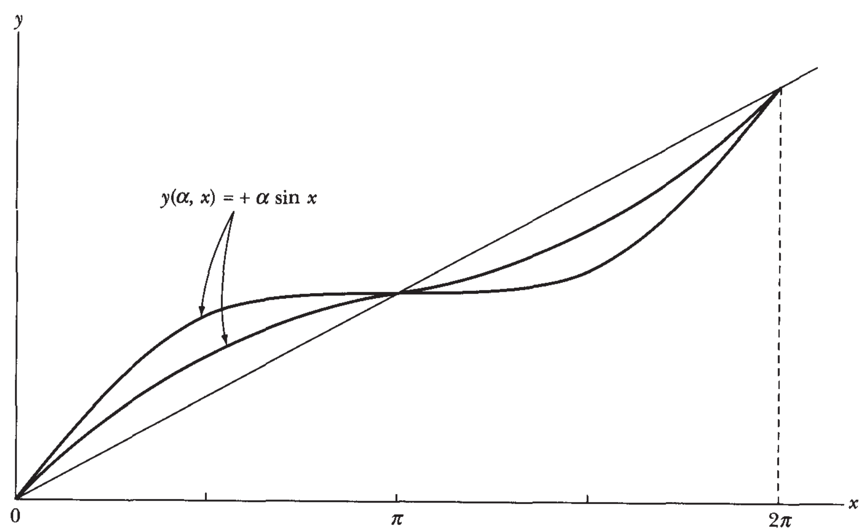

FIGURE 6-2 Example 6.1. The various paths $y(\alpha, x) = x + \alpha \sin x$. The extremum path occurs for $\alpha = 0$.

then

$$f = \left(\frac{dy(\alpha, x)}{dx}\right)^2 = 1 + 2\alpha \cos x + \alpha^2 \cos^2 x \tag{6.8}$$

Equation 6.3 now becomes

$$J(\alpha) = \int_0^{2\pi} (1 + 2\alpha \cos x + \alpha^2 \cos^2 x)\, dx \tag{6.9}$$

$$= 2\pi + \alpha^2 \pi \tag{6.10}$$

Thus we see the value of $J(\alpha)$ is always greater than $J(0)$, no matter what value (positive or negative) we choose for α. The condition of Equation 6.4 is also satisfied.

6.3 Euler's Equation

To determine the result of the condition expressed by Equation 6.4, we perform the indicated differentiation in Equation 6.3:

$$\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha} \int_{x_1}^{x_2} f\{y, y'; x\}\, dx \tag{6.11}$$

Because the limits of integration are fixed, the differential operation affects only the integrand. Hence,

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}\right) dx \tag{6.12}$$

From Equation 6.2, we have

$$\frac{\partial y}{\partial \alpha} = \eta(x); \qquad \frac{\partial y'}{\partial \alpha} = \frac{d\eta}{dx} \tag{6.13}$$

Equation 6.12 becomes

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\frac{d\eta}{dx}\right) dx \tag{6.14}$$

The second term in the integrand can be integrated by parts:

$$\int u\, dv = uv - \int v\, du \tag{6.15}$$

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\frac{d\eta}{dx}\, dx = \frac{\partial f}{\partial y'}\eta(x)\Bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)\, dx \tag{6.16}$$

The integrated term vanishes because $\eta(x_1) = \eta(x_2) = 0$. Therefore, Equation 6.12 becomes

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\eta(x) - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)\right] dx$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\eta(x)\, dx \tag{6.17}$$

The integral in Equation 6.17 now appears to be independent of α. But the functions y and y' with respect to which the derivatives of f are taken are still functions of α. Because $(\partial J/\partial \alpha)|_{\alpha=0}$ must vanish for the extremum value and because $\eta(x)$ is an arbitrary function (subject to the conditions already stated), the integrand in Equation 6.17 must itself vanish for $\alpha = 0$:

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0} \qquad \text{Euler's equation} \tag{6.18}$$

where now y and y' are the original functions, independent of α. This result is known as **Euler's equation,*** which is a necessary condition for J to have an extremum value.

EXAMPLE 6.2

We can use the calculus of variations to solve a classic problem in the history of physics: the *brachistochrone.*† Consider a particle moving in a constant force field starting at rest from some point (x_1, y_1) to some lower point (x_2, y_2). Find the path that allows the particle to accomplish the transit in the least possible time.

Solution. The coordinate system may be chosen so that the point (x_1, y_1) is at the origin. Further, let the force field be directed along the positive x-axis as in Figure 6-3. Because the force on the particle is constant—and if we ignore the possibility of friction—the field is conservative, and the total energy of the particle is $T + U =$ const. If we measure the potential from the point $x = 0$ [i.e., $U(x = 0) = 0$], then, because the particle starts from rest, $T + U = 0$. The kinetic energy is $T = \frac{1}{2}mv^2$, and the potential energy is $U = -Fx = -mgx$, where g is the acceleration imparted by the force. Thus

$$v = \sqrt{2gx} \tag{6.19}$$

The time required for the particle to make the transit from the origin to (x_2, y_2) is

$$t = \int_{(x_1,y_1)}^{(x_2,y_2)} \frac{ds}{v} = \int \frac{(dx^2 + dy^2)^{1/2}}{(2gx)^{1/2}}$$

$$= \int_{x_1=0}^{x_2} \left(\frac{1 + y'^2}{2gx}\right)^{1/2} dx \tag{6.20}$$

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*Derived first by Euler in 1744. When applied to mechanical systems, this is known as the *Euler-Lagrange equation.*

†First solved by Johann Bernoulli (1667–1748) in 1696.

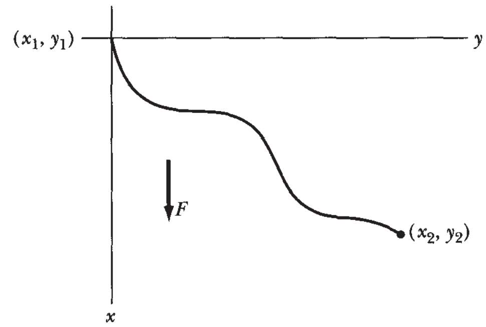

FIGURE 6-3 Example 6.2. The *brachistochrone* problem is to find the path of a particle moving from (x_1, y_1) to (x_2, y_2) that occurs in the least possible time. The force field acting on the particle is F, which is down and constant.

The time of transit is the quantity for which a minimum is desired. Because the constant $(2g)^{-1/2}$ does not affect the final equation, the function f may be identified as

$$f = \left(\frac{1 + y'^2}{x}\right)^{1/2} \tag{6.21}$$

And, because $\partial f/\partial y = 0$, the Euler equation (Equation 6.18) becomes

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

or

$$\frac{\partial f}{\partial y'} = \text{constant} \equiv (2a)^{-1/2}$$

where a is a new constant.

Performing the differentiation $\partial f/\partial y'$ on Equation 6.21 and squaring the result, we have

$$\frac{y'^2}{x(1 + y'^2)} = \frac{1}{2a} \tag{6.22}$$

This may be put in the form

$$y = \int \frac{x\,dx}{(2ax - x^2)^{1/2}} \tag{6.23}$$

We now make the following change of variable:

$$\begin{aligned} x &= a(1 - \cos\theta) \\ dx &= a \sin\theta\, d\theta \end{aligned} \tag{6.24}$$

The integral in Equation 6.23 then becomes

$$y = \int a(1 - \cos\theta)\, d\theta$$

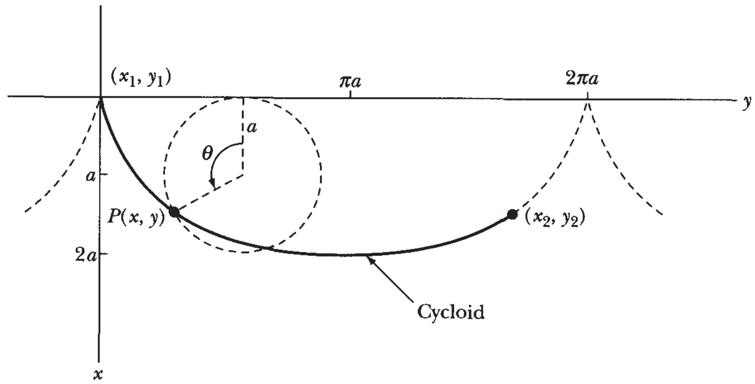

FIGURE 6-4 Example 6.2. The solution of the brachistochrone problem is a cycloid.

and

$$y = a(\theta - \sin\theta) + \text{constant} \tag{6.25}$$

The parametric equations for a *cycloid** passing through the origin are

$$\left.\begin{aligned} x &= a(1 - \cos\theta) \\ y &= a(\theta - \sin\theta) \end{aligned}\right\} \tag{6.26}$$

which is just the solution found, with the constant of integration set equal to zero to conform with the requirement that (0, 0) is the starting point of the motion. The path is then as shown in Figure 6-4, and the constant a must be adjusted to allow the cycloid to pass through the specified point (x_2, y_2). Solving the problem of the brachistochrone does indeed yield a path the particle traverses in a *minimum* time. But the procedures of variational calculus are designed only to produce an extremum—either a minimum or a maximum. It is almost always the case in dynamics that we desire (and find) a minimum for the problem.

EXAMPLE 6.3

Consider the surface generated by revolving a line connecting two fixed points (x_1, y_1) and (x_2, y_2) about an axis coplanar with the two points. Find the equation of the line connecting the points such that the surface area generated by the revolution (i.e., the area of the surface of revolution) is a minimum.

Solution. We assume that the curve passing through (x_1, y_1) and (x_2, y_2) is revolved about the y-axis, coplanar with the two points. To calculate the total area of the surface of revolution, we first find the area dA of a strip. Refer to Figure 6-5.

*A cycloid is a curve traced by a point on a circle rolling on a plane along a line in the plane. See the dashed sphere rolling along $x = 0$ in Figure 6-4.

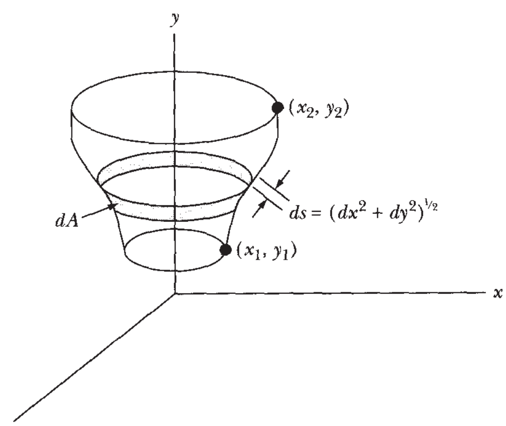

FIGURE 6-5 Example 6.3. The geometry of the problem and area *dA* are indicated to minimize the surface of revolution around the *y*-axis.

$$dA = 2\pi x\, ds = 2\pi x(dx^2 + dy^2)^{1/2} \tag{6.27}$$

$$A = 2\pi \int_{x_1}^{x_2} x(1 + y'^2)^{1/2}\, dx \tag{6.28}$$

where $y' = dy/dx$. To find the extremum value we let

$$f = x(1 + y'^2)^{1/2} \tag{6.29}$$

and insert into Equation 6.18:

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial y'} = \frac{xy'}{(1 + y'^2)^{1/2}}$$

therefore,

$$\frac{d}{dx}\left[\frac{xy'}{(1 + y'^2)^{1/2}}\right] = 0$$

$$\frac{xy'}{(1 + y'^2)^{1/2}} = \text{constant} \equiv a \tag{6.30}$$

From Equation 6.30, we determine

$$y' = \frac{a}{(x^2 - a^2)^{1/2}} \tag{6.31}$$

$$y = \int \frac{a\, dx}{(x^2 - a^2)^{1/2}} \tag{6.32}$$

The solution of this integration is

$$y = a \cos \mathrm{h}^{-1}\left(\frac{x}{a}\right) + b \tag{6.33}$$

where a and b are constants of integration determined by requiring the curve to pass through the points (x_1, y_1) and (x_2, y_2). Equation 6.33 can also be written as

$$x = a \cos \mathrm{h}\left(\frac{y - b}{a}\right) \tag{6.34}$$

which is more easily recognized as the equation of a *catenary*, the curve of a flexible cord hanging freely between two points of support.

Choose two points located at (x_1, y_1) and (x_2, y_2) joined by a curve $y(x)$. We want to find $y(x)$ such that if we revolve the curve around the x-axis, the surface area of the revolution is a minimum. This is the "soap film" problem, because a soap film suspended between two wire circular rings takes this shape (Figure 6-6). We want to minimize the integral of the area $dA = 2\pi y\, ds$ where $ds = \sqrt{1 + y'^2}\, dx$ and $y' = dy/dx$.

$$A = 2\pi \int_{x_1}^{x_2} y\sqrt{1 + y'^2}dx \tag{6.35}$$

We find the extremum by setting $f = y\sqrt{1 + y'^2}$ and inserting into Equation 6.18. The derivatives we need are

$$\frac{\partial f}{\partial y} = \sqrt{1 + y'^2}$$

$$\frac{\partial f}{\partial y'} = \frac{yy'}{\sqrt{1 + y'^2}}$$

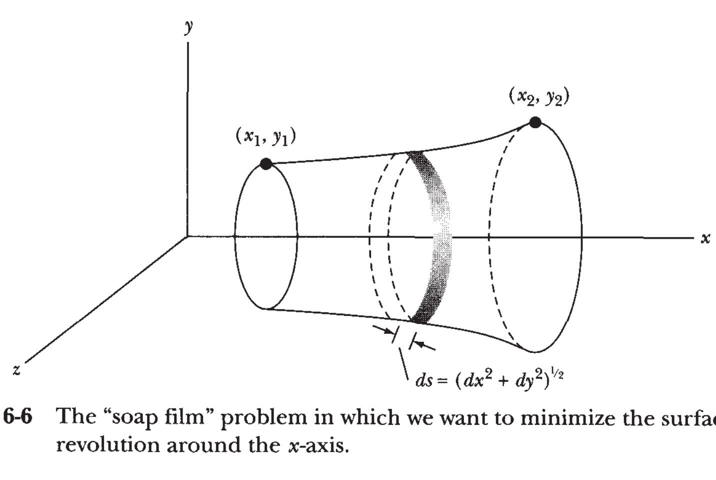

FIGURE 6-6 The "soap film" problem in which we want to minimize the surface area of revolution around the x-axis.

Equation 6.18 becomes

$$\sqrt{1+y'^2} = \frac{d}{dx}\left[\frac{yy'}{\sqrt{1+y'^2}}\right] \tag{6.36}$$

Equation 6.36 does not appear to be a simple equation to solve for $y(x)$. Let's stop and think about whether there might be an easier method of solution. You may have noticed that this problem is just like Example 6.3, but in that case we were minimizing a surface of revolution about the y-axis rather than around the x-axis. The solution to the soap film problem should be identical to Equation 6.34 if we interchange x and y. But how did we end up with such a complicated equation as Equation 6.36? We blindly chose x as the independent variable and decided to find the function $y(x)$. In fact, in general, we can choose the independent variable to be anything we want: x, θ, t, or even y. If we choose y as the independent variable, we would need to interchange x and y in many of the previous equations that led up to Euler's equation (Equation 6.18). It might be easier in the beginning to just interchange the variables that we started with (i.e., call the horizontal axis y in Figure 6-6 and let the independent variable be x). (In a right-handed coordinate system, the x-direction would be down, but that presents no difficulty in this case because of symmetry.) No matter what we do, the solution of our present problem would just parallel Example 6.3. Unfortunately, it is not always possible to look ahead to make the best choice of independent variable. Sometimes we just have to proceed by trial and error.

6.4 The "Second Form" of the Euler Equation

A second equation may be derived from Euler's equation that is convenient for functions that do not explicitly depend on x: $\partial f/\partial x = 0$. We first note that for any function $f(y, y'; x)$ the derivative is a sum of terms

$$\frac{df}{dx} = \frac{d}{dx}f\{y, y'; x\} = \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} + \frac{\partial f}{\partial x}$$

$$= y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} + \frac{\partial f}{\partial x} \tag{6.37}$$

Also

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = y''\frac{\partial f}{\partial y'} + y'\frac{d}{dx}\frac{\partial f}{\partial y'}$$

or, substituting from Equation 6.37 for $y''(\partial f/\partial y')$,

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = \frac{df}{dx} - \frac{\partial f}{\partial x} - y'\frac{\partial f}{\partial y} + y'\frac{d}{dx}\frac{\partial f}{\partial y'} \tag{6.38}$$

The last two terms in Equation 6.38 may be written as

$$y'\left(\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y}\right)$$

which vanishes in view of the Euler equation (Equation 6.18). Therefore,

$$\boxed{\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0} \tag{6.39}$$

We can use this so-called "second form" of the Euler equation in cases in which f does not depend explicitly on x, and $\partial f/\partial x = 0$. Then,

$$f - y'\frac{\partial f}{\partial y'} = \text{constant} \qquad \left(\text{for } \frac{\partial f}{\partial x} = 0\right) \tag{6.40}$$

EXAMPLE 6.4

A *geodesic* is a line that represents the shortest path between any two points when the path is restricted to a particular surface. Find the geodesic on a sphere.

Solution. The element of length on the surface of a sphere of radius ρ is given (see Equation F.15 with $dr = 0$) by

$$ds = \rho(d\theta^2 + \sin^2\theta\, d\phi^2)^{1/2} \tag{6.41}$$

The distance s between points 1 and 2 is therefore

$$s = \rho\int_1^2\left[\left(\frac{d\theta}{d\phi}\right)^2 + \sin^2\theta\right]^{1/2} d\phi \tag{6.42}$$

and, if s is to be a minimum, f is identified as

$$f = (\theta'^2 + \sin^2\theta)^{1/2} \tag{6.43}$$

where $\theta' \equiv d\theta/d\phi$. Because $\partial f/\partial\phi = 0$, we may use the second form of the Euler equation (Equation 6.40), which yields

$$(\theta'^2 + \sin^2\theta)^{1/2} - \theta' \cdot \frac{\partial}{\partial\theta'}(\theta'^2 + \sin^2\theta)^{1/2} = \text{constant} \equiv a \tag{6.44}$$

Differentiating and multiplying through by f, we have

$$\sin^2\theta = a(\theta'^2 + \sin^2\theta)^{1/2} \tag{6.45}$$

This may be solved for $d\phi/d\theta = \theta'^{-1}$, with the result

$$\frac{d\phi}{d\theta} = \frac{a\csc^2\theta}{(1 - a^2\csc^2\theta)^{1/2}} \tag{6.46}$$

Solving for ϕ, we obtain

$$\phi = \sin^{-1}\left(\frac{\cot\theta}{\beta}\right) + \alpha \tag{6.47}$$

where α is the constant of integration and $\beta^2 \equiv (1 - a^2)/a^2$. Rewriting Equation 6.47 produces

$$\cot\theta = \beta \sin(\phi - \alpha) \tag{6.48}$$

To interpret this result, we convert the equation to rectangular coordinates by multiplying through by $\rho \sin\theta$ to obtain, on expanding $\sin(\phi - \alpha)$,

$$(\beta \cos\alpha)\rho \sin\theta \sin\phi - (\beta \sin\alpha)\rho \sin\theta \cos\phi = \rho\cos\theta \tag{6.49}$$

Because α and β are constants, we may write them as

$$\beta\cos\alpha \equiv A, \qquad \beta \sin\alpha \equiv B \tag{6.50}$$

Then Equation 6.49 becomes

$$A(\rho \sin\theta \sin\phi) - B(\rho\sin\theta\cos\phi) = (\rho\cos\theta) \tag{6.51}$$

The quantities in the parentheses are just the expressions for y, x, and z, respectively, in spherical coordinates (see Figure F-3, Appendix F); therefore Equation 6.51 may be written as

$$Ay - Bx = z \tag{6.52}$$

which is the equation of a plane passing through the center of the sphere. Hence the geodesic on a sphere is the path that the plane forms at the intersection with the surface of the sphere—a *great circle*. Note that the great circle is the maximum as well as the minimum "straight-line" distance between two points on the surface of a sphere.

6.5 Functions with Several Dependent Variables

The Euler equation derived in the preceding section is the solution of the variational problem in which it was desired to find the single function $y(x)$ such that the integral of the functional f was an extremum. The case more commonly encountered in mechanics is that in which f is a functional of several dependent variables:

$$f = f\{y_1(x), y_1'(x), y_2(x), y_2'(x), \cdots; x\} \tag{6.53}$$

or simply

$$f = f\{y_i(x), y_i'(x); x\}, \quad i = 1, 2, \cdots, n \tag{6.54}$$

In analogy with Equation 6.2, we write

$$y_i(\alpha, x) = y_i(0, x) + \alpha\eta_i(x) \tag{6.55}$$

The development proceeds analogously (cf. Equation 6.17), resulting in

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \sum_i \left(\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} \right) \eta_i(x)\, dx \tag{6.56}$$

Because the individual variations—the $\eta_i(x)$—are all independent, the vanishing of Equation 6.56 when evaluated at $\alpha = 0$ requires the separate vanishing of *each* expression in the brackets:

$$\boxed{\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} = 0, \qquad i = 1, 2, \cdots, n} \tag{6.57}$$

6.6 Euler's Equations When Auxiliary Conditions Are Imposed

Suppose we want to find, for example, the shortest path between two points on a surface. Then, in addition to the conditions already discussed, there is the condition that the path must satisfy the equation of the surface, say, $g\{y_i; x\} = 0$. Such an equation was implicit in the solution of Example 6.4 for the geodesic on a sphere where the condition was

$$g = \sum_i x_i^2 - \rho^2 = 0 \tag{6.58}$$

that is,

$$r = \rho = \text{constant} \tag{6.59}$$

But in the general case, we must make explicit use of the auxiliary equation or equations. These equations are also called **equations of constraint.** Consider the case in which

$$f = f\{y_i, y_i'; x\} = f\{y, y', z, z'; x\} \tag{6.60}$$

The equation corresponding to Equation 6.17 for the case of *two* variables is

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right) \frac{\partial y}{\partial \alpha} + \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'} \right) \frac{\partial z}{\partial \alpha} \right] dx \tag{6.61}$$

But now there also exists an equation of constraint of the form

$$g\{y_i; x\} = g\{y, z; x\} = 0 \tag{6.62}$$

and the variations $\partial y/\partial \alpha$ and $\partial z/\partial \alpha$ are no longer independent, so the expressions in parentheses in Equation 6.61 do not separately vanish at $\alpha = 0$.

Differentiating g from Equation 6.62, we have

$$dg = \left(\frac{\partial g}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial \alpha} \right) d\alpha = 0 \tag{6.63}$$

where no term in x appears since $\partial x/\partial\alpha = 0$. Now

$$\left.\begin{aligned} y(\alpha, x) &= y(x) + \alpha\eta_1(x) \\ z(\alpha, x) &= z(x) + \alpha\eta_2(x) \end{aligned}\right\} \tag{6.64}$$

Therefore, by determining $\partial y/\partial\alpha$ and $\partial z/\partial\alpha$ from Equation 6.64 and inserting into the term in parentheses of Equation 6.63, which, in general, must be zero, we obtain

$$\frac{\partial g}{\partial y}\eta_1(x) = -\frac{\partial g}{\partial z}\eta_2(x) \tag{6.65}$$

Equation 6.61 becomes

$$\frac{\partial J}{\partial\alpha} = \int_{x_1}^{x_2}\left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\eta_1(x) + \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'}\right)\eta_2(x)\right]dx$$

Factoring $\eta_1(x)$ out of the square brackets and writing Equation 6.65 as

$$\frac{\eta_2(x)}{\eta_1(x)} = -\frac{\partial g/\partial y}{\partial g/\partial z}$$

we have

$$\frac{\partial J}{\partial\alpha} = \int_{x_1}^{x_2}\left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right) - \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'}\right)\left(\frac{\partial g/\partial y}{\partial g/\partial z}\right)\right]\eta_1(x)\,dx \tag{6.66}$$

This latter equation now contains the single arbitrary function $\eta_1(x)$, which is not in any way restricted by Equation 6.64, and on requiring the condition of Equation 6.4, the expression in the brackets must vanish. Thus we have

$$\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\left(\frac{\partial g}{\partial y}\right)^{-1} = \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'}\right)\left(\frac{\partial g}{\partial z}\right)^{-1} \tag{6.67}$$

The left-hand side of this equation involves only derivatives of f and g with respect to y and y', and the right-hand side involves only derivatives with respect to z and z'. Because y and z are both functions of x, the two sides of Equation 6.67 may be set equal to a function of x, which we write as $-\lambda(x)$:

$$\left.\begin{aligned} \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \lambda(x)\frac{\partial g}{\partial y} &= 0 \\ \frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'} + \lambda(x)\frac{\partial g}{\partial z} &= 0 \end{aligned}\right\} \tag{6.68}$$

The complete solution to the problem now depends on finding *three* functions: $y(x)$, $z(x)$, and $\lambda(x)$. But there are *three* relations that may be used: the two equations (Equation 6.68) and the equation of constraint (Equation 6.62). Thus, there is a sufficient number of relations to allow a complete solution. Note that here $\lambda(x)$ is considered to be *undetermined** and is obtained as a part of the solution. The function $\lambda(x)$ is known as a **Lagrange undetermined multiplier.**

*The function $\lambda(x)$ was introduced in Lagrange's *Mécanique analytique* (Paris, 1788).

For the general case of several dependent variables and several auxiliary conditions, we have the following set of equations:

$$\boxed{\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} + \sum_j \lambda_j(x)\frac{\partial g_j}{\partial y_i} = 0} \tag{6.69}$$

$$\boxed{g_j\{y_i; x\} = 0} \tag{6.70}$$

If $i = 1, 2, \ldots, m$, and $j = 1, 2, \ldots, n$, Equation 6.69 represents m equations in $m + n$ unknowns, but there are also the n equations of constraint (Equation 6.70). Thus, there are $m + n$ equations in $m + n$ unknowns, and the system is soluble.

Equation 6.70 is equivalent to the set of n differential equations

$$\sum_i \frac{\partial g_j}{\partial y_i}\, dy_i = 0, \qquad \begin{cases} i = 1, 2, \cdots, m \\ j = 1, 2, \cdots, n \end{cases} \tag{6.71}$$

In problems in mechanics, the constraint equations are frequently differential equations rather than algebraic equations. Therefore, equations such as Equation 6.71 are sometimes more useful than the equations represented by Equation 6.70. (See Section 7.5 for an amplification of this point.)

EXAMPLE 6.5

Consider a disk rolling without slipping on an inclined plane (Figure 6-7). Determine the equation of constraint in terms of the "coordinates"* y and θ.

Solution. The relation between the coordinates (which are not independent) is

$$y = R\theta \tag{6.72}$$

where R is the radius of the disk. Hence the equation of constraint is

$$g(y, \theta) = y - R\theta = 0 \tag{6.73}$$

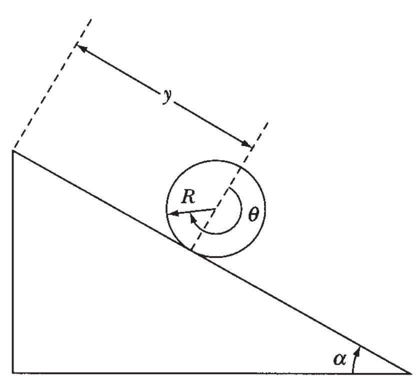

FIGURE 6-7 Example 6.5. A disk rolls down an inclined plane without slipping.

*These are actually the *generalized coordinates* discussed in Section 7.3; see also Example 7.9.

and

$$\frac{\partial g}{\partial y} = 1, \quad \frac{\partial g}{\partial \theta} = -R \tag{6.74}$$

are the quantities associated with λ, the single undetermined multiplier for this case.

The constraint equation can also appear in an integral form. Consider the isoperimetric problem that is stated as finding the curve $y = y(x)$ for which the functional

$$J[y] = \int_a^b f\{y, y'; x\} dx \tag{6.75}$$

has an extremum, and the curve $y(x)$ satisfies boundary conditions $y(a) = A$ and $y(b) = B$ as well as the second functional

$$K[y] = \int_a^b g\{y, y'; x\} dx \tag{6.76}$$

that has a fixed value for the length of the curve (ℓ). This second functional represents an integral constraint.

Similarly to what we have done previously,* there will be a constant λ such that $y(x)$ is the extremal solution of the functional

$$\int_a^b (f + \lambda g)\, dx. \tag{6.77}$$

The curve $y(x)$ then will satisfy the differential equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \lambda\left(\frac{\partial g}{\partial y} - \frac{d}{dx}\frac{\partial g}{\partial y'}\right) = 0 \tag{6.78}$$

subject to the constraints $y(a) = A$, $y(b) = B$, and $K[y] = \ell$. We will work an example for this so-called *Dido Problem*.†

EXAMPLE 6.6

One version of the Dido Problem is to find the curve $y(x)$ of length ℓ bounded by the x-axis on the bottom that passes through the points $(-a, 0)$ and $(a, 0)$ and encloses the largest area. The value of the endpoints a is determined by the problem.

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*For a proof, see Ge63, p. 43.

†The isoperimetric problem was made famous by Virgil's poem *Aeneid*, which described Queen Dido of Carthage, who in 900 B.C. was given by a local king as much land as she could enclose with an ox's hide. In order to maximize her claim, she had the hide cut into thin strips and tied them end to end. She apparently knew enough mathematics to know that for a perimeter of a given length, the maximum area enclosed is a circle.

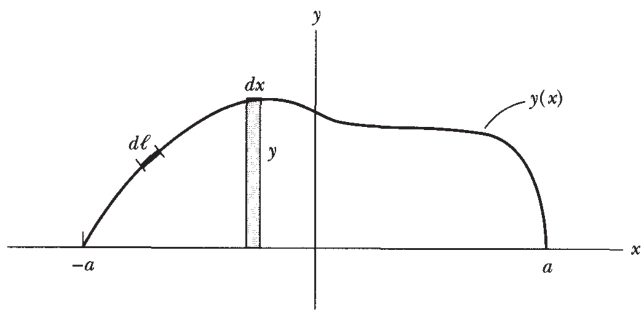

FIGURE 6-8 Example 6.6. We want to find the curve $y(x)$ that maximizes the area above the $y = 0$ line consistent with a fixed perimeter length. The curve must go through $x = -a$ and a. The differential area $dA = ydx$, and the differential length along the curve is $d\ell$.

Solution. We can use the equations just developed to solve this problem. We show in Figure 6-8 that the differential area $dA = y\,dx$. We want to maximize the area, so we want to find the extremum solution for Equation 6.75, which becomes

$$J = \int_{-a}^{a} y\,dx \tag{6.79}$$

The constraint equations are

$$y(x): y(-a) = 0, y(a) = 0 \quad \text{and} \quad K = \int d\ell = \ell. \tag{6.80}$$

The differential length along the curve $d\ell = (dx^2 + dy^2)^{1/2} = (1 + y'^2)^{1/2}\,dx$ where $y' = dy/dx$. The constraint functional becomes

$$K = \int_{-a}^{a} [1 + y'^2]^{1/2} dx = \ell. \tag{6.81}$$

We now have $y(x) = y$ and $g(x) = \sqrt{1 + y'^2}$, and we use these functions in Equation 6.78.

$$\frac{\partial f}{\partial y} = 1, \quad \frac{\partial f}{\partial y'} = 0, \quad \frac{\partial g}{\partial y} = 0, \quad \frac{\partial g}{\partial y'} = \frac{y'}{(1 + y'^2)^{1/2}}$$

Equation 6.78 becomes

$$1 - \lambda \frac{d}{dx}\left[\frac{y'}{(1 + y'^2)^{1/2}}\right] = 0 \tag{6.82}$$

We manipulate Equation 6.82 to find

$$\frac{d}{dx}\left[\frac{y'}{(1 + y'^2)^{1/2}}\right] = \frac{1}{\lambda} \tag{6.83}$$

We integrate over x to find

$$\frac{\lambda y'}{\sqrt{(1 + y'^2)}} = x - C_1$$

where C_1 is an integration constant. This can be rearranged to be

$$dy = \frac{\pm(x - C_1)\,dx}{\sqrt{\lambda^2 - (x - C_1)^2}}$$

This equation is integrated to find

$$y = \mp\sqrt{\lambda^2 - (x - C_1)^2} + C_2 \tag{6.84}$$

where C_2 is another integration constant. We can rewrite this as the equation of a circle of radius λ.

$$(x - C_1)^2 + (y - C_2)^2 = \lambda^2 \tag{6.85}$$

The maximum area is a semicircle bounded by the $y = 0$ line. The semicircle must go through (x, y) points of $(-a, 0)$ and $(a, 0)$, which means the circle must be centered at the origin, so that $C_1 = 0 = C_2$, and the radius $= a = \lambda$. The perimeter of the top half of the semicircle is what we called ℓ, and the perimeter length of a half circle is πa. Therefore, we have $\pi a = \ell$, and $a = \ell/\pi$.

6.7 The δ Notation

In analyses that use the calculus of variations, we customarily use a shorthand notation to represent the variation. Thus, Equation 6.17, which can be written as

$$\frac{\partial J}{\partial \alpha}\,d\alpha = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\frac{\partial y}{\partial \alpha}\,d\alpha\,dx \tag{6.86}$$

may be expressed as

$$\delta J = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\delta y\,dx \tag{6.87}$$

where

$$\left.\begin{aligned} \frac{\partial J}{\partial \alpha}d\alpha &\equiv \delta J \\ \frac{\partial y}{\partial \alpha}d\alpha &\equiv \delta y \end{aligned}\right\} \tag{6.88}$$

The condition of extremum then becomes

$$\delta J = \delta\int_{x_1}^{x_2} f\{y, y'; x\}dx = 0 \tag{6.89}$$

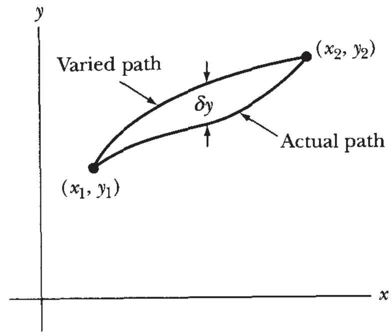

FIGURE 6-9 The varied path is a virtual displacement δy from the actual path consistent with all the forces and constraints.

Taking the variation symbol δ inside the integral (because, by hypothesis, the limits of integration are not affected by the variation), we have

$$\begin{aligned}\delta J &= \int_{x_1}^{x_2} \delta f \, dx \\ &= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' \right) dx\end{aligned} \quad \textbf{(6.90)}$$

But

$$\delta y' = \delta \left(\frac{dy}{dx} \right) = \frac{d}{dx}(\delta y) \quad \textbf{(6.91)}$$

so

$$\delta J = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \frac{d}{dx} \delta y \right) dx \quad \textbf{(6.92)}$$

Integrating the second term by parts as before, we find

$$\delta J = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \delta y \, dx \quad \textbf{(6.93)}$$

Because the variation δy is arbitrary, the extremum condition $\delta J = 0$ requires the integrand to vanish, thereby yielding the Euler equation (Equation 6.18).

Although the δ notation is frequently used, it is important to realize that it is only a shorthand expression of the more precise differential quantities. The varied path represented by δy can be thought of physically as a virtual displacement from the actual path consistent with all the forces and constraints (see Figure 6-9). This variation δy is distinguished from an actual differential displacement dy by the condition that $dt = 0$—that is, that time is fixed. The varied path δy, in fact, need not even correspond to a possible path of motion. The variation must vanish at the endpoints.

PROBLEMS

6-1. Consider the line connecting $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (1, 1)$. Show explicitly that the function $y(x) = x$ produces a minimum path length by using the varied function $y(\alpha, x) = x + \alpha \sin \pi(1 - x)$. Use the first few terms in the expansion of the resulting elliptic integral to show the equivalent of Equation 6.4.

6-2. Show that the shortest distance between two points on a plane is a straight line.

6-3. Show that the shortest distance between two points in (three-dimensional) space is a straight line.

6-4. Show that the geodesic on the surface of a right circular cylinder is a segment of a helix.

6-5. Consider the surface generated by revolving a line connecting two fixed points (x_1, y_1) and (x_2, y_2) about an axis coplanar with the two points. Find the equation of the line connecting the points such that the surface area generated by the revolution (i.e., the area of the surface of revolution) is a minimum. Obtain the solution by using Equation 6.39.

6-6. Reexamine the problem of the brachistochrone (Example 6.2) and show that the time required for a particle to move (frictionlessly) to the *minimum* point of the cycloid is $\pi\sqrt{a/g}$, *independent* of the starting point.

6-7. Consider light passing from one medium with index of refraction n_1 into another medium with index of refraction n_2 (Figure 6-A). Use Fermat's principle to minimize time, and derive the law of refraction: $n_1 \sin \theta_1 = n_2 \sin \theta_2$.

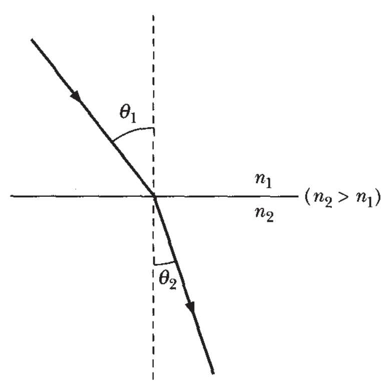

FIGURE 6-A Problem 6-7.

6-8. Find the dimensions of the parallelepiped of maximum volume circumscribed by **(a)** a sphere of radius R; **(b)** an ellipsoid with semiaxes a, b, c.

6-9. Find an expression involving the function $\phi(x_1, x_2, x_3)$ that has a minimum average value of the square of its gradient within a certain volume V of space.

6-10. Find the ratio of the radius R to the height H of a right-circular cylinder of fixed volume V that minimizes the surface area A.

6-11. A disk of radius R rolls without slipping inside the parabola $y = ax^2$. Find the equation of constraint. Express the condition that allows the disk to roll so that it contacts the parabola at one and only one point, independent of its position.

6-12. Repeat Example 6.4, finding the shortest path between any two points on the surface of a sphere, but use the method of the Euler equations with an auxiliary condition imposed.

6-13. Repeat Example 6.6 but do not use the constraint that the $y = 0$ line is the bottom part of the area. Show that the plane curve of a given length, which encloses a maximum area, is a circle.

6-14. Find the shortest path between the (x, y, z) points $(0, -1, 0)$ and $(0, 1, 0)$ on the conical surface $z = 1 - \sqrt{x^2 + y^2}$. What is the length of the path? Note: this is the shortest mountain path around a volcano.

6-15. **(a)** Find the curve $y(x)$ that passes through the endpoints $(0, 0)$ and $(1, 1)$ and minimizes the functional $I[y] = \int_0^1[(dy/dx)^2 - y^2]dx$. **(b)** What is the minimum value of the integral? **(c)** Evaluate $I[y]$ for a straight line $y = x$ between the points $(0, 0)$ and $(1, 1)$.

6-16. **(a)** What curve on the surface $z = x^{3/2}$ joining the points $(x, y, z) = (0, 0, 0)$ and $(1, 1, 1)$ has the shortest arc length? **(b)** Use a computer to produce a plot showing the surface and the shortest curve on a single plot.

6-17. The corners of a rectangle lie on the ellipse $(x/a)^2 + (y/b)^2 = 1$. **(a)** Where should the corners be located in order to maximize the area of the rectangle? **(b)** What fraction of the area of the ellipse is covered by the rectangle with maximum area?

6-18. A particle of mass m is constrained to move under gravity with no friction on the surface $xy = z$. What is the trajectory of the particle if it starts from rest at $(x, y, z) = (1, -1, -1)$ with the z-axis vertical?